

Thermodynamic phase transition in the Dicke model for multi-level systems

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1977 J. Phys. A: Math. Gen. 10 L131

(<http://iopscience.iop.org/0305-4470/10/7/002>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 129.252.86.83

The article was downloaded on 30/05/2010 at 14:01

Please note that [terms and conditions apply](#).

LETTER TO THE EDITOR

Thermodynamic phase transition in the Dicke model for multi-level systems

R Gilmore

Physics Department, University of South Florida, Tampa, Florida 33620, USA

Received 5 April 1977

Abstract. Thermodynamic critical behaviour between excited states of a multi-level system is characterised by four distinct values of the appropriate coupling constant λ .

A second-order phase transition occurs in the 2-level Dicke model provided the coupling constant λ exceeds a critical value λ_c determined by $\lambda_c^2/\epsilon\omega = 1$. Here $\epsilon = \epsilon_2 - \epsilon_1$ is the energy difference between the upper and lower atomic states, and ω is the energy of the approximately resonant photon mode. Throughout we take $\hbar = 1$, λ real and $\lambda > 0$.

In the 3-level extension of the Dicke model, the thermodynamic critical properties may or may not be similar. We assume the atom-field interaction occurs between levels 1 and 2. If the third level has energy exceeding ϵ_1 , the critical properties are as described in the opening paragraph. If the third level has energy ϵ_0 , $\epsilon_0 < \epsilon_1 < \epsilon_2$, then the thermodynamic critical properties are characterised by four critical values $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ of the coupling constant λ .

These four values can be determined from the free energy per particle of the interacting N -atom, single field mode system. This in turn can be estimated using field coherent states (Wang and Hioe 1973), atomic coherent states (Hepp and Lieb 1973), the coupled order parameter method (Gilmore and Bowden 1976a), or a bifurcation analysis (Gilmore and Bowden 1976b). It is convenient to introduce the order parameter $x = \langle a/\sqrt{N} \rangle$, where a is the photon annihilation operator. The free energy F/N is determined by (Gilmore 1977)

$$\begin{aligned} F/N &= \min_{0 \leq x < \infty} \Phi(\beta, x) + O(1/N) \\ \Phi(\beta, x) &= \omega x^2 - \beta^{-1} \ln [e^{-\beta\epsilon_0} + e^{-\beta(\epsilon_1 + \epsilon_2)/2} (e^{\beta\theta} + e^{-\beta\theta})] \\ \theta^2 &= \left(\frac{\epsilon_2 - \epsilon_1}{2} \right)^2 + \lambda^2 x^2. \end{aligned} \tag{1}$$

In figure 1 we plot those values of $x(\beta)$ for which $\Phi(\beta, x)$ is stationary, for several values of λ . The potential $\Phi(\beta, x)$ always has a stationary value for $x(\beta) = 0$. This solution is called the disordered branch. The primary branches $x(\beta) \neq 0$ may be unstable, meta-stable, or stable, as indicated.

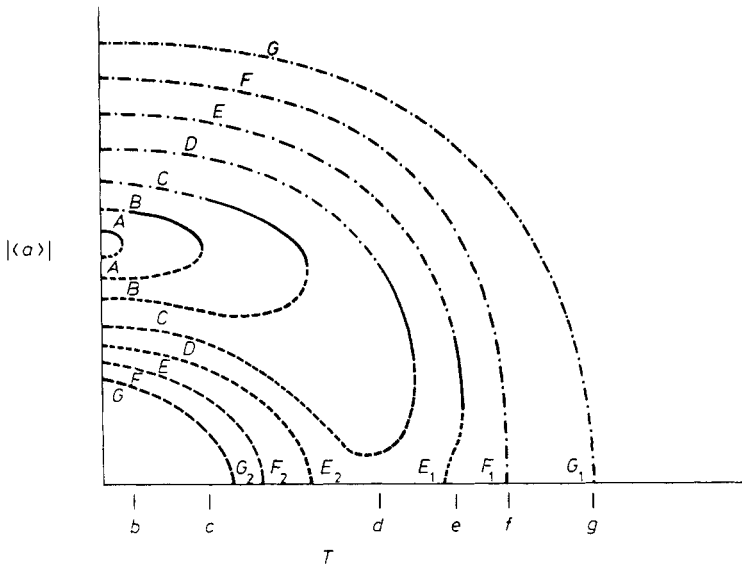


Figure 1. Values of the order parameter $x(\beta)$ for which the potential $\Phi(\beta, x)$ is stationary, for seven values of λ . For primary branches: the chain curve denotes stable branches, the full curve denotes metastable branches and the broken curve denotes unstable branches.

The four critical values of λ are determined by searching for the smallest value of λ for which:

- λ_1 : $\Phi(\beta, x)$ has a metastable state at $T = 0, x \neq 0$;
- λ_2 : $\Phi(\beta, x)$ has a stable state at $T = 0, x \neq 0$;
- λ_3 : $\Phi(\beta, x)$ has only unstable bifurcating solutions; and
- λ_4 : $\Phi(\beta, x)$ has a stable bifurcating solution.

The critical values λ_1 and λ_2 are determined by searching for solutions $x \neq 0$ of $\partial\Phi(\beta, x)/\partial x = 0$ in the $T \rightarrow 0$ limit. Such solutions exist for $\Lambda^2 > \Lambda_1^2 = 1 + [2(\epsilon_1 - \epsilon_0)/(\epsilon_2 - \epsilon_1)]$ where $\Lambda^2 = \lambda^2/\omega(\epsilon_2 - \epsilon_1)$. For $\lambda > \lambda_1$ there are two non-zero solutions, one unstable, the other metastable until $\lambda = \lambda_2$ determined by $\Lambda_2 - \Lambda_2^{-1} = 2[(\epsilon_1 - \epsilon_0)/(\epsilon_2 - \epsilon_1)]^{1/2}$.

The critical values λ_3 and λ_4 are determined by expanding $\Phi(\beta, x)$ in powers of x , and looking for changes of stability along the disordered branch $x = 0$. To fourth order this expansion is

$$\begin{aligned} \Phi(\beta, x) &= -\beta^{-1} \ln z(\beta) + C_2(\beta)x^2 + C_4(\beta)x^4 \\ z(\beta) &= e^{-\beta\epsilon_0} + e^{-\beta\epsilon_1} + e^{-\beta\epsilon_2} \\ C_2(\beta) &= \omega - \frac{\lambda^2}{(\epsilon_2 - \epsilon_1)} \frac{e^{-\beta\epsilon_1} - e^{-\beta\epsilon_2}}{z(\beta)} \end{aligned} \tag{2}$$

$$C_4(\beta) = \frac{\lambda^4}{(\epsilon_2 - \epsilon_1)^3} \left[\frac{e^{-\beta\epsilon_1} - e^{-\beta\epsilon_2}}{z(\beta)} + \frac{1}{2}\beta(\epsilon_2 - \epsilon_1) \left(\frac{e^{-\beta\epsilon_1} - e^{-\beta\epsilon_2}}{z(\beta)} \right)^2 - \beta(\epsilon_2 - \epsilon_1) \frac{e^{-\beta\epsilon_1} + e^{-\beta\epsilon_2}}{z(\beta)} \right].$$

Primary branches bifurcate from the disordered branch when $C_2(\beta) = 0$. The smallest

value $\lambda = \lambda_3$ for which bifurcation occurs is determined from $\max_{T \in [0, \infty)} \Lambda_3^2[(e^{-\beta \epsilon_1} - e^{-\beta \epsilon_2})/z(\beta)] = 1$. The stability of the bifurcating solutions is determined by the sign of $C_4(\beta)$ at the bifurcation points. This is always negative (unstable) for the branches bifurcating at lower temperature (figure 1, E_2, F_2, G_2). It is also negative for λ in the range $\lambda_3 \leq \lambda < \lambda_4$. The value λ_4 is determined by setting $C_4(\beta_t) = 0$, where β_t is the higher temperature solution of $C_2(\beta) = 0$.

The critical properties of the 3-level system with coupling between the two excited levels are determined by the value of λ as follows (figure 1).

$\lambda < \lambda_1$. $\langle a/\sqrt{N} \rangle = 0$ at all temperatures.

$\lambda_1 < \lambda < \lambda_2$ (curve A, $\lambda = \lambda_A$). At sufficiently low temperature a metastable ordered state exists.

$\lambda_2 < \lambda < \lambda_3$ (curves B, C, D, $\lambda_B < \lambda_C < \lambda_D$). At low temperature an ordered state exists. For λ_B the ordered state is metastable to the right of the mark b on the horizontal axis of figure 1. To the left, the disordered state is metastable. A first-order phase transition occurs at this point, and hysteresis may occur. The transition temperature increases as λ increases. These first-order transitions are not surrounded by spinodal points.

$\lambda_3 < \lambda < \lambda_4$ (curve E, $\lambda = \lambda_E$). Two primary branches bifurcate from the disordered branch. The low temperature branch E_2 is unstable. The high temperature branch E_1 is initially unstable, with T increasing as x increases. However, this branch eventually turns around, becoming metastable, then stable, as x increases further and T decreases towards zero. A first-order phase transition occurs at e . Unlike the transitions at b, c , and d , this transition is surrounded by two spinodal points where $dT/dx = 0$ for x on branch E_1 . These points are the bifurcation and turn-around points. The disordered branch is stable for T greater than e , metastable for T between e and the E_1 bifurcation point, unstable between the E_2 and E_1 bifurcation points, and metastable again below the E_2 bifurcation point. The range of values of λ for which spinodal points occur is very small because $(\lambda_4 - \lambda_3)/\lambda_4 \ll 1$.

$\lambda = \lambda_4$ (curve F, $\lambda = \lambda_F = \lambda_4$). The bifurcation point $\lambda = \lambda_4$, $\beta = \beta_t$, $x_{F_1} = 0$ is a tricritical point.

$\lambda_4 < \lambda$ (curve G, $\lambda = \lambda_G$). The low temperature bifurcating branch G_2 is unstable, while the high temperature branch is stable. The disordered branch is stable above the bifurcation point for G_1 , unstable between the bifurcation points of G_2 and G_1 , and metastable below the G_2 bifurcation. A second-order phase transition occurs at the higher solution of $C_2(\beta) = 0$.

The thermodynamic critical properties for r -level systems are qualitatively the same as those for 3-level systems. Let $\epsilon_1 < \epsilon_2 < \dots < \epsilon_r$ be the ordered non-degenerate levels of the atomic systems, with ω_{ji} the energy of the near resonant photon mode connecting levels j and i ($j > i$). The free energy for the appropriate Dicke Hamiltonian (Gilmore and Bowden 1976b, equation (7.7)) can be computed by the methods outlined above. The results are as follows, with $\Lambda_{ji}^2 = |\lambda_{ji}|^2/\omega_{ji}(\epsilon_j - \epsilon_i)$ and $z(\beta) = \sum_{k=1}^r e^{-\beta \epsilon_k}$.

$j > i = 1$. A second-order phase transition occurs if $\Lambda_{j1} > 1$. The critical temperature T_c is determined by

$$\Lambda_{j1}^2 \frac{e^{-\beta_c \epsilon_1} - e^{-\beta_c \epsilon_j}}{z(\beta_c)} = 1. \tag{3}$$

The associated order-parameter behaviour as a function of temperature is essentially as given in Gilmore and Bowden (1976a, figure 1).

$j > i > 1$. The four critical values of Λ_{ji} are given by

$$\begin{aligned} (\Lambda_{ji})_1^2 &= \frac{1 + \gamma_{ji}}{1 - \gamma_{ji}} \\ (\Lambda_{ji})_2^2 &= \frac{1 + (\gamma_{ji})^{1/2}}{1 - (\gamma_{ji})^{1/2}}; & \gamma_{ji} &= \frac{\epsilon_i - \epsilon_1}{\epsilon_j - \epsilon_1} \\ \max_{\beta > 0} (\Lambda_{ji})_3^2 \frac{e^{-\beta\epsilon_i} - e^{-\beta\epsilon_j}}{z(\beta)} &= 1 \end{aligned} \quad (4a)$$

and $(\Lambda_{ji})_4$ is determined by the simultaneous vanishing of $C_2(\beta)$ and $C_4(\beta)$ (tricritical point):

$$\begin{aligned} C_2(\beta) &\approx 1 - (\Lambda_{ji})_4^2 \left(\frac{e^{-\beta\epsilon_i} - e^{-\beta\epsilon_j}}{z(\beta)} \right) \\ C_4(\beta) &\approx \frac{e^{-\beta\epsilon_i} - e^{-\beta\epsilon_j}}{z(\beta)} + \frac{\beta(\epsilon_j - \epsilon_i)}{2} \left(\frac{e^{-\beta\epsilon_i} - e^{-\beta\epsilon_j}}{z(\beta)} \right)^2 - \beta(\epsilon_j - \epsilon_i) \frac{e^{-\beta\epsilon_i} + e^{-\beta\epsilon_j}}{z(\beta)}. \end{aligned} \quad (4b)$$

The associated order-parameter behaviour as a function of temperature is essentially as given in figure 1, and the critical properties are as discussed above.

It is a pleasure to thank C U Hogle for aid with numerical computations.

References

- Gilmore R 1977 *J. Math. Phys.* **18** 17-22
 Gilmore R and Bowden C M 1976a *Phys. Rev. A* **13** 1898-907
 — 1976b *J. Math. Phys.* **17** 1617-25
 Hepp K and Lieb E H 1973 *Phys. Rev. A* **8** 2517-25
 Wang Y K and Hioe F T 1973 *Phys. Rev. A* **7** 831-6